

Particles motion on topological Lifshitz black holes in 3+1 dimensions

Marco Olivares • Germán Rojas •
Yerko Vásquez • J. R. Villanueva

Contents

1	Introduction	1
2	Topological Lifshitz Black Hole	2
3	Geodesic Structure	3
3.1	Null Geodesics	3
3.1.1	Radial Motion	3
3.1.2	Angular Motion	4
3.2	Time-Like Geodesics	5
3.2.1	Radial Motion	5
3.2.2	Angular motion	6
4	Summary	6

Marco Olivares

Instituto de Física,
Pontificia Universidad Católica de Valparaíso,
Av. Universidad 330, Curauma,
Valparaíso, Chile.
e-mail: marco.olivaresrubilar@gmail.com

Germán Rojas

North American College, Av. 21 de Mayo 833,
Arica, Chile.
e-mail: spanw17@gmail.com

Yerko Vásquez

Departamento de Ciencias Físicas,
Facultad de Ingeniería, Ciencias y Administración,
Universidad de La Frontera,
Avenida Francisco Salazar 01145, Casilla 54-D,
Temuco, Chile.
e-mail: yvasquez@ufro.cl

J. R. Villanueva

Departamento de Física y Astronomía,
Universidad de Valparaíso,
Gran Bretaña 1111, Playa Ancha,
Valparaíso, Chile.

Centro de Astrofísica de Valparaíso,
Gran Bretaña 1111, Playa Ancha,
Valparaíso, Chile.

e-mail: jose.villanuevalob@dfa.uv.cl

Abstract In the present paper we study the causal structure of a topological black hole presented by Mann R. B. JHEP **06**, 075 (2009) by mean the standard Lagrangian procedure, which allow us analyze qualitatively the behavior of test particles using the effective potential. Then, the geodesic motion of massive and massless particles is obtained analytically. We find that confined orbits are forbidden on this spacetime, however radial photons can escape to infinity in an infinite proper time but in a finite coordinate time, this correspond to an interesting and novel result.

Keywords Lifshitz black holes; Geodesics; Causal structure.

1 Introduction

In the last years much attention have been focused in the application of the AdS/CFT correspondence (Maldacena 1998) beyond high energy physics to another areas of physics, where a more general class of spacetimes could be the gravitational dual, for instance, to non relativistic scale invariant theories of condensed matter physics. In (Kachru et al. 2008) has been conjectured gravity duals of non relativistic Lifshitz -like fixed points which describe multicritical points in certain magnetic materials and liquid crystal, these curved duals are called Lifshitz spacetimes. An important feature of these spacetimes is its invariance under an anisotropic scale transformation. There are many theories with anisotropic invariant scale of interest in studying such critical points instead of the scale invariance which arises in the conformal group, particularly in the studies of critical exponent theory and phase transitions. Systems with these features have appears in the description of strongly correlated electrons of strange

metals (Hartnoll 2010). The scale invariance is expressed as $t \rightarrow \lambda^z t$, $x \rightarrow \lambda x$, where $z \neq 1$ is the critical exponent which measure the degree of anisotropy between spatial and temporal scalings.

On the gravitational side, D -dimensional spacetimes that exhibit these symmetries are described by the Lifshitz metrics

$$ds^2 = -\frac{r^{2z}}{\ell^{2z}} dt^2 + \frac{l^2}{r^2} dr^2 + \frac{r^2}{l^2} d\vec{x}^2, \quad (1)$$

where \vec{x} represents a $D - 2$ dimensional spatial vector and l denotes the length scale in the geometry. If $z = 1$, the spacetime is the usual anti-de Sitter metric in Poincaré coordinates and have the larger symmetry $SO(D - 1, 2)$. All curvature invariants of metric (1) are constant and these spacetimes have a null curvature singularity at $r \rightarrow 0$ for $z \neq 1$, this can be seen by computing the tidal forces between infalling particles. This singularity is reached in finite proper time by infalling observers so the spacetime is geodesically incomplete (Horowitz & Way 2011). A natural extension of the above spacetime is consider black hole solutions in this background whose asymptotic behavior is given by (1), these are called asymptotically Lifshitz black holes (Balasubramanian & McGreevy 2009)-(Dehghani & Mann 2010).

We are interesting in analyze the geodesic structure of a class of asymptotically Lifshitz black hole recently found in the literature, performing a study of freely moving of test particles and photons. The geodesic equations of motions are a set of ordinary differential equations describing the evolution of the coordinates of these test particles as a function of an affine parameter. If the spacetime have symmetries the conserved quantities associate to these symmetries simplify the analysis of the equations. The geodesic structure of the Schwarzschild, Reissner-Nordström and Kerr black holes were studied by Chandrasekhar (Chandrasekar 1983). The geodesic study in a Schwarzschild space-time background was the key to understand various astrophysical phenomena, as planetary motion, gravitational lensing and radar delay among others. Besides of astrophysical objects it is important to study, motivated by the AdS/CFT correspondence (Maldacena 1998) and its generalizations, the geodesic structure of black holes with an asymptotic behavior different than the flat case, furthermore these black holes provide a theoretical laboratory for understanding interesting features of black holes physics. In this regard, geodesics around the Schwarzschild anti-de Sitter black hole was studied in (Kraniotis & Whitehouse 2003)-(Cruz et al. 2005)-(Hackmann & Lammerzahl 2008)-(Hackmann & Lammerzahl 2008), and the Schwarzschild

de Sitter case in (Jaklitsch et al. 1989)-(Stuchlík & Calvani 1991)-(Podolsky 1999). The motion of uncharged particles in Reissner-Nordström black hole with a non-zero cosmological constant has been studied in (Stuchlík & Hledík 2002)-(Villanueva et al. 2013), while the study of charged particles can be found in (Olivares et al. 2011). Orbits in Kerr and Kerr (anti) de Sitter are calculated in (Kraniotis 2004). Study of geodesic motion of test particles in higher dimensional Schwarzschild, Schwarzschild anti-de Sitter, Reissner-Nordström and Reissner-Nordström anti-de Sitter spacetimes can be found in references (Hackmann et al. 2009)-(Gibbons & Vyska 2012), where complete solutions and a classification of the possible orbits in these geometries in term of elliptic functions have been obtained.

In this article we will study the geodesic structure of the topological Lifshitz black hole in 3+1 dimensions with critical exponent $z = 2$ (Mann 2009). We will perform an analysis of particle motions by means of an effective potential, we will provide graphics of the orbits and classify different kinds of motion for particles by the values of its angular momentum.

This paper is organized as follows. In Sec. 2, we briefly review some features of topological Lifshitz black hole. In Sec. 3 we obtain the radial equation of motion for geodesics with an effective potential. Therefore we study null and time-like geodesics. We conclude in Sec. 4 with some final remarks.

2 Topological Lifshitz Black Hole

Four dimensional topological Lifshitz black hole was presented in (Mann 2009) where the author obtained a black hole solution with critical exponent $z = 2$, which posses an event horizon if the curvature of the transverse spatial sections is negative $k = -1$. These kinds of black holes are described by the following metric,

$$ds^2 = -\frac{r^2 f(r)}{\ell^2} dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sinh^2\theta d\phi^2), \quad (2)$$

where the lapse function $f(r)$ is given by

$$f(r) = \frac{r^2}{\ell^2} - \frac{1}{2}, \quad (3)$$

and the coordinates are defined in the intervals $-\infty < t < \infty$, $r > 0$, $0 \leq \phi \leq 2\pi$, and $0 \leq \theta \leq 2\pi$. In FIG. 1 we show a graph of the lapse function for topological Lifshitz black holes.

The event horizon is located at $r_+ = \ell/\sqrt{2}$, and corresponds to a coordinate singularity, which can be seen from the Ricci scalar,

$$R = \frac{1}{r^2} - \frac{11}{r_+^2}, \quad (4)$$

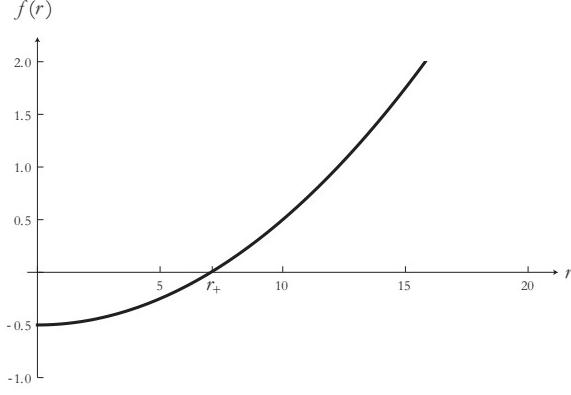


Fig. 1 Plot of the lapse function as a function of the radial coordinate, r , with $\ell = 10$ ($r_+ = 5\sqrt{2}$).

such that,

$$R_{r=r_+} = -\frac{10}{r_+^2}. \quad (5)$$

The principal quadratic invariant of the Ricci tensor and the Kretschmann scalar are given by the following expressions

$$R_{\mu\nu}R^{\mu\nu} = \frac{33r^4 - 8r_+^2r^2 + r_+^4}{r_+^4r^4}, \quad (6)$$

and

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{3(r_+^4 - 2r_+^2r^2 + 9r^4)}{r_+^4r^4}, \quad (7)$$

respectively. These invariants tell us that the physical singularity is located at $r = 0$.

In (Gonzalez et al. 2012) a study of the quasinormal modes and the absorption cross section for this black hole was performed, and was shown that it is stable under scalar field perturbations.

Finally, it's interesting to notice that metric (2) looks analogue to a zero-mass topological AdS black hole (Mann 1997).

3 Geodesic Structure

In order to compute the geodesic structure of the topological Lifshitz black hole, we will use the standard Lagrangian procedure (Cruz et al. 2005), and thus, we write the Lagrangian associated to the metric (2), resulting

$$2\mathcal{L} = -\frac{r^2 f(r)}{2r_+^2}t^2 + \frac{\dot{r}^2}{f(r)} + r^2(\dot{\theta}^2 + \sinh^2\theta\dot{\phi}^2) = -m, \quad (8)$$

where the dot corresponds to derivative with respect to an affine parameter along the geodesic, and, by normalization, $m = 1$ for massive particles and $m = 0$ for massless particles.

Since (t, ϕ) are cyclic coordinates, their conjugate momenta, Π_q , are conserved. In our case we obtain,

$$\Pi_\phi = r^2\dot{\phi}\sinh^2\theta = L, \quad (9)$$

and,

$$\Pi_t = -\frac{r^2}{2r_+^2}f(r)\dot{t} = -\sqrt{E}. \quad (10)$$

Here L is the angular momentum, but the constant of motion E cannot be associated globally to the energy because this spacetime is not asymptotically flat. On the other hand, eq. (9) implies that the motion is performed in an invariant plane, which, for simplicity, we choose to be the plane defined by $\theta = \theta_0$, such that, $\sinh\theta_0 = 1$. In this way, we rewrite eq. (9) as

$$r^2\dot{\phi} = L. \quad (11)$$

Then, introducing eqs. (10) and (11) into eq. (8), we obtain the radial equation of motion,

$$\dot{r}^2 = 2\frac{r_+^2}{r^2}[E - V(r)], \quad (12)$$

where $V(r)$ is the effective potential given by

$$V(r) = \frac{1}{4}\frac{r^2}{r_+^2}\left(\frac{r^2}{r_+^2} - 1\right)\left(m + \frac{L^2}{r^2}\right). \quad (13)$$

In the next sections, based in this effective potential, we will study all possible motion for massive and massless particles.

3.1 Null Geodesics

Null geodesic corresponds to a photon trajectory, i. e., $m = 0$, so, the effective potential, (13), takes the form,

$$V_n(r) = \frac{1}{4}\frac{L^2}{r_+^2}\left(\frac{r^2}{r_+^2} - 1\right). \quad (14)$$

3.1.1 Radial Motion

Radial photons are characterized by vanishing angular momentum ($L = 0$), which implies that effective potential vanishes too,

$$V_{nr}(r) = 0, \quad (15)$$

and thus, photons have the possibility of not falling into the black hole and escape indefinitely towards spatial infinity. Therefore, eq. (12) becomes

$$\dot{r}^2 = 2E \frac{r_+^2}{r^2}, \quad (16)$$

so, an elementary integration yields to

$$\tau(r) = \pm \frac{1}{2} \frac{R_0^2}{\sqrt{2E} r_+} \left[\left(\frac{r}{R_0} \right)^2 - 1 \right], \quad (17)$$

where R_0 corresponds to the radial distance in which $\tau = 0$.

Next, making use of the identity

$$\frac{dr}{d\tau} = \left(\frac{dr}{dt} \right) t, \quad (18)$$

together with eqs. (10) and (16), we obtain the quadrature

$$\frac{dr}{dt} = \pm \frac{1}{2\sqrt{2}} \frac{r}{r_+} \left(\frac{r^2}{r_+^2} - 1 \right). \quad (19)$$

Therefore, integrating this last equation from R_0 ($t(r = R_0) = 0$) to r , we obtain

$$t(r) = \pm \sqrt{2} r_+ \left[\ln \left| \frac{r^2 - r_+^2}{R_0^2 - r_+^2} \right| - \ln \left(\frac{r}{R_0} \right)^2 \right]. \quad (20)$$

In FIG. 2 we show the proper time τ and the coordi-

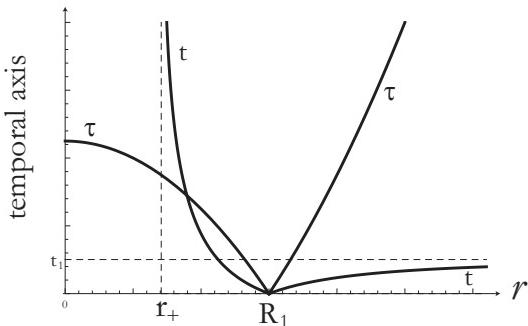


Fig. 2 Plot of the proper time, $\tau(r)$, and coordinate time, $t(r)$, as a function of the radial coordinate r with $\ell = 10$.

nate time t as a function of the radial coordinate r . As in Schwarzschild spacetime, photons fall into the black hole in a finite proper time, however an external observer see that photons fall asymptotically to the black hole. Moreover, the topological Lifshitz black hole admits radial photons come to infinity, and, in the proper system they come to infinity in infinity proper time, but an external observer see that they come to infinity in a

finite coordinate time, t_1 , given by

$$t_1 = \lim_{r \rightarrow \infty} t(r) = \sqrt{2} r_+ \ln \left| \frac{R_0^2}{R_0^2 - r_+^2} \right|. \quad (21)$$

This situation is unique with respect to the Einstein's spacetimes.

3.1.2 Angular Motion

In this case photons posses angular momentum $L > 0$, thus, its effective potential is

$$V_{na}(r) = \frac{1}{4} \frac{L^2}{r_+^2} \left(\frac{r^2}{r_+^2} - 1 \right), \quad (22)$$

which is shown in FIG. 3.

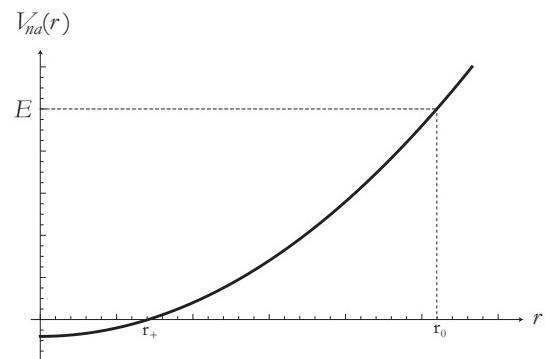


Fig. 3 Effective potential for photons in angular motion, $V_{na}(r)$, with $r_+ = 5\sqrt{2}$ ($\ell = 10$) and $L = 4$.

Next, using the identity

$$\dot{r} = \left(\frac{dr}{d\phi} \right) \dot{\phi}, \quad (23)$$

together with the motion integral (9), eq. (11) can be written as

$$\left(\pm \frac{dr}{d\phi} \right)^2 = 2 r_+^2 r^2 \left(\frac{1}{\mathfrak{D}^2} - \frac{1}{4} \frac{r^2}{r_+^4} \right), \quad (24)$$

where $\mathfrak{D} = \left(\frac{1}{b^2} + \frac{1}{4} \frac{1}{r_+^2} \right)^{-1/2}$ is the *anomalous impact parameter*, and $b = \sqrt{L^2/E}$ corresponds to the usual impact parameter. Now, making the change of variable $x = \frac{r}{\sqrt{2} r_+}$, we obtain

$$\phi(x) = \int_{x_0}^x \frac{-dx}{x \cdot \sqrt{\mathcal{P}(x)}}, \quad (25)$$

where $\mathcal{P}(x)$ is a second order polynomial given by $\mathcal{P}(x) = x_0^2 - x^2$, with $x_0 = \frac{r_0}{\sqrt{2} r_+}$, and r_0 corresponds

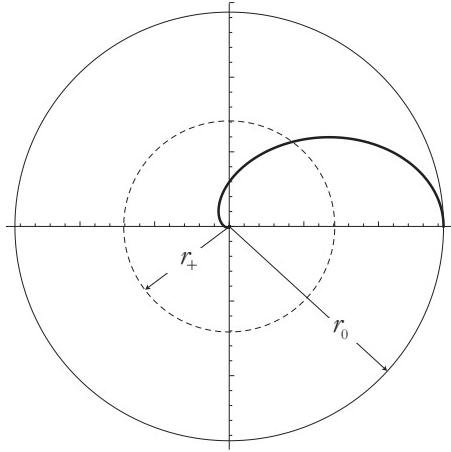


Fig. 4 Polar trajectory (27) with $r_+ = 5\sqrt{2}$ ($\ell = 10$) and $L = 4$.

to the value of r when $t = \tau = 0$. Integrating eq. (25), we obtain

$$\phi(x) = \frac{1}{x_0} \operatorname{arccosh} \left(\frac{x_0}{x} \right), \quad (26)$$

therefore, the polar trajectory is given by

$$r(\phi) = r_0 \operatorname{sech} \left(\frac{r_0}{\sqrt{2} r_+} \phi \right), \quad (27)$$

which is shown in FIG. 4.

3.2 Time-Like Geodesics

In this section we will consider the motion of massive particles on the background of the topological Lifshitz black hole. In this case we have $m = 1$, therefore, the effective potential (12) acquires the form

$$V_t(r) = \frac{1}{4} \frac{r^2}{r_+^2} \left(\frac{r^2}{r_+^2} - 1 \right) \left(1 + \frac{L^2}{r^2} \right). \quad (28)$$

3.2.1 Radial Motion

For this situation the particles have zero angular momentum $L = 0$, and the effective potential (28) simplifies to

$$V_{tr}(r) = \frac{1}{4} \frac{r^2}{r_+^2} \left(\frac{r^2}{r_+^2} - 1 \right), \quad (29)$$

this potential is shown in FIG. 5.

In this way, eqs. (11) and (29) lead to the quadrature

$$\left(\frac{dr}{d\tau} \right)^2 = 2 \frac{r_+^2}{r^2} \left[E - \frac{1}{4} \frac{r^2}{r_+^2} \left(\frac{r^2}{r_+^2} - 1 \right) \right], \quad (30)$$

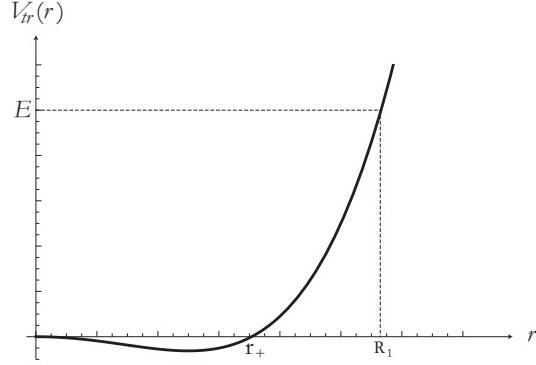


Fig. 5 Effective potential for massive radial particles, $V_{tr}(r)$, with $r_+ = 5\sqrt{2}$ ($\ell = 10$) and $L = 4$.

so that making the change of variable $r = r_+ \sqrt{2y}$, eq. (30) becomes

$$\tau(y) = -\frac{r_+}{\sqrt{2}} \int_{y_0}^y \frac{dy}{\sqrt{E + \frac{y}{2} - y^2}}, \quad (31)$$

where we assume that $\tau(y_0) \equiv \tau_0 = 0$. So, integrating eq. (31) we find

$$\sqrt{2} \frac{\tau(r)}{r_+} = \frac{\pi}{2} - \arcsin \left(\frac{2r^2 - r_+^2}{2R_1^2 - r_+^2} \right), \quad (32)$$

and therefore, we obtain the final expression

$$r(\tau) = \frac{r_+}{\sqrt{2}} \sqrt{1 + \left(2 \frac{R_1^2}{r_+^2} - 1 \right) \sin \left(\frac{\pi}{2} - \sqrt{2} \frac{\tau}{r_+} \right)}. \quad (33)$$

On the other side, using (10) and (18) into eq. (30), and then integrating the quadrature, we obtain

$$\begin{aligned} \frac{t(r)}{\sqrt{2} r_+} &= \operatorname{arccosh} \left[\frac{8E + \frac{r^2}{r_+^2} - 1}{\left(\frac{r^2}{r_+^2} - 1 \right) \left(2 \frac{R_1^2}{r_+^2} - 1 \right)} \right] + \\ &- \operatorname{arccosh} \left[\frac{8E + \frac{r^2}{r_+^2}}{\frac{r^2}{r_+^2} \left(2 \frac{R_1^2}{r_+^2} - 1 \right)} \right] + \varpi_t, \end{aligned} \quad (34)$$

where

$$\begin{aligned} \varpi_t &= \operatorname{arccosh} \left[\frac{8E + \frac{R_1^2}{r_+^2}}{\frac{R_1^2}{r_+^2} \left(2 \frac{R_1^2}{r_+^2} - 1 \right)} \right] + \\ &- \operatorname{arccosh} \left[\frac{8E + \frac{R_1^2}{r_+^2} - 1}{\left(\frac{R_1^2}{r_+^2} - 1 \right) \left(2 \frac{R_1^2}{r_+^2} - 1 \right)} \right], \end{aligned} \quad (35)$$

and the turning point, R_1 , is given by

$$R_1 = \sqrt{\frac{1 + \sqrt{1 + 16E}}{2}} r_+. \quad (36)$$

FIG. 6 shows a graph of eqs. (33) and (34).

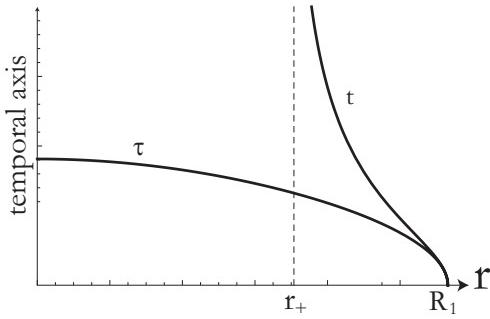


Fig. 6 Proper and coordinate time for radial massive particles as a function of radial coordinate r with $r_+ = 5\sqrt{2}$ ($\ell = 10$) and $L = 4$.

3.2.2 Angular motion

Particles with angular motion are characterized by $L > 0$. The effective potential in this case is given by

$$V_{ta}(r) = \frac{1}{4} \frac{r^2}{r_+^2} \left(\frac{r^2}{r_+^2} - 1 \right) \left(1 + \frac{L^2}{r^2} \right), \quad (37)$$

which is depicted in FIG. 7.

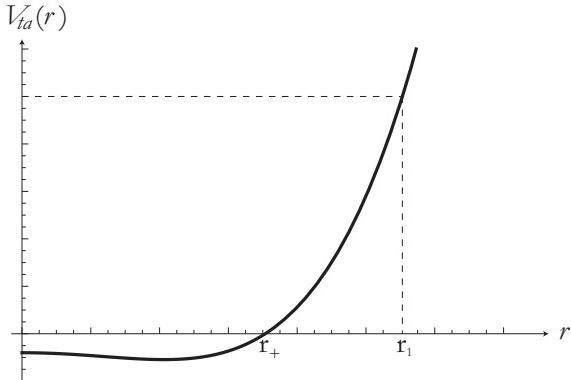


Fig. 7 Effective potential of angular massive particles, $V_{ta}(r)$, with $r_+ = 5\sqrt{2}$ ($\ell = 10$) and $L = 4$.

Using eqs. (9), (21) and (37) we get

$$\left(\frac{dr}{d\phi} \right)^2 = 2 \frac{r_+^2 r^2}{L^2} \left[E - \frac{1}{4} \frac{r^2}{r_+^2} \left(\frac{r^2}{r_+^2} - 1 \right) \left(1 + \frac{L^2}{r^2} \right) \right], \quad (38)$$

and, after the change of variable $r = r_+ \sqrt{2y}$ in the above equation, we obtain the following expression

$$\phi(x) = -\frac{1}{4} \frac{L^2}{r_+^2} \int_{y_0}^y \frac{dy}{y \sqrt{\mu - \sigma y - y^2}}, \quad (39)$$

where

$$\mu = E + \frac{1}{8} \frac{L^4}{r_+^4}, \quad \text{and} \quad \sigma = \frac{1}{4} \frac{L^4}{r_+^4} - \frac{1}{2}. \quad (40)$$

Making the integration and returning to the original variables we obtain an analytical expression for $r(\phi)$, the polar form of the orbit

$$r(\phi) = r_1 \sqrt{\frac{1 + \Omega}{1 + \Omega \cosh(\alpha_0 \phi)}}, \quad (41)$$

where the turning point is given explicitly by

$$r_1 = \frac{2\sqrt{\mu/\sigma} r_+}{\sqrt{1 + \Omega}}, \quad (42)$$

and the constants Ω and α_0 are defined as

$$\Omega = \sqrt{1 + 4 \frac{\mu}{\sigma^2}}, \quad \text{and} \quad \alpha_0 = \frac{4\sqrt{\mu} r_+^2}{L^2}, \quad (43)$$

respectively. FIG. 8 shows the polar trajectory of massive particles (41).

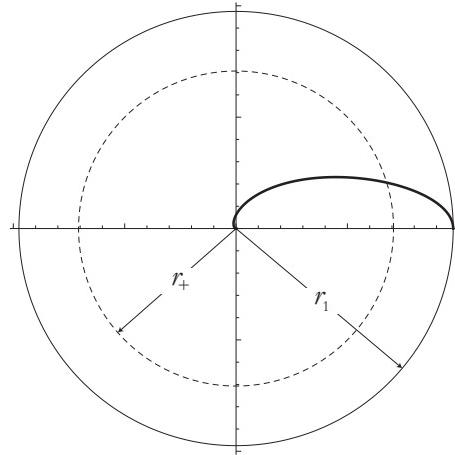


Fig. 8 Polar trajectory (41) with $r_+ = 5\sqrt{2}$ ($\ell = 10$) and $L = 4$.

4 Summary

In this work we have studied the causal structure of topological Lifshitz black hole using the standard Lagrangian procedure. We analyzed the radial and angular motion of massless and massive test particles on this background. Because the metric coefficients doesn't depends on the temporal and azimuthal coordinates, it allows us to find two integrals of motion, and reduce the system of equations to one radial equation of motion with an effective potential. We have obtained analytical expressions for the proper time and coordinate time

as a function of the radial coordinate as well as analytical expressions for the polar form of the orbits in all cases considered. For radial photons we obtain that the solutions show a similar behavior to what occurs in the Schwarzschild black hole, in the sense that in the proper time framework, these radial massless particles reach the event horizon and cross it in a finite proper time, however in the coordinate time framework, the external observer sees that takes an infinite time the particles to reach the horizon. Besides, on other hand, topological Lifshitz black hole admits radial photons come to infinity in a finite coordinate time as seen by an external observer, these results are depicted in FIG. 2. For photons with angular momentum, the effective potential is unbounded and increases with the radial distance acting as an attractive force, as show FIG. 3. Therefore, non radial photons always will reach the event horizon, this can be seen additionally in FIG. 4 where the polar trajectory of the photon is illustrated. For massive particles the effective potential have a qualitatively similar behavior for both radial and non-radial particles. Analogous to the above case of non-radial photons, these potentials increase with the radial distance, which means that the particles are confined and all them will reach the event horizon. Therefore the space-time doesn't admit bounded orbits. In FIG 8 we show the orbit of a generic non-radial massive particle.

Acknowledgements Y. V. is supported by FONDECYT grant 11121148. M. O. thanks to PUCV.

References

- Balasubramanian K. and McGreevy J.: An Analytic Lifshitz black hole. *Phys. Rev. D* **80**, 104039 (2009).
- Chandrasekhar S.: The Mathematical Theory of Black Holes. Oxford University Press, New York (1983).
- Cruz N., Olivares M. and Villanueva J. R.: The geodesic structure of the Schwarzschild anti-de Sitter Black Hole. *Class. Quantum Grav.* **22**, 1167-1190 (2005).
- Dehghani M. H. and Mann R. B.: Lovelock-Lifshitz Black Holes. *JHEP* **1007**, 019 (2010).
- Gibbons G. W. and Vyska M.: The Application of Weierstrass elliptic functions to Schwarzschild Null Geodesics. *Class. Quant. Grav.* **29**, 065016 (2012).
- Gonzalez P. A., Moncada F. and Vasquez Y.: Quasinormal Modes, Stability Analysis and Absorption Cross Section for 4-dimensional Topological Lifshitz Black Hole. *Eur. Phys. J. C* **72**, 2255 (2012).
- Hackmann E., Kagramanova V., Kunz J. and Lammerzahl C.: Analytical solutions of the geodesic equation in higher dimensions. *Phys. Rev. D* **78**, 124018 (2008), Erratum-*ibid.* **79**, 029901 (2009).
- Hackmann E. and Lammerzahl C.: Complete Analytic Solution of the Geodesic Equation in Schwarzschild(Anti)-de Sitter Spacetimes. *Phys. Rev. Lett.* **100**, 171101 (2008).
- Hackmann E. and Lammerzahl C.: Geodesic equation in Schwarzschild-(anti-)de Sitter space-times: Analytical solutions and applications. *Phys. Rev. D* **78**, 024035 (2008).
- Hartnoll S. A., Polchinski J., Silverstein E. and Tong D.: Towards strange metallic holography. *JHEP* **1004**, 120 (2010).
- Horowitz G. T. and Way B.: Lifshitz Singularities. *Phys. Rev. D* **85**, 046008 (2012).
- Jaklitsch M. J., Hellaby C. and Matravers D.R.: Particle motion in the spherically symmetric vacuum solution with positive cosmological constant. *Gen. Rel. Grav.* **21**, 941 (1989).
- Kachru S., Liu X. and Mulligan M.: Gravity Duals of Lifshitz-like Fixed Points. *Phys. Rev. D* **78**, 106005 (2008).
- Kraniotis G. V. and Whitehouse S. B.: Compact calculation of the Perihelion Precession of Mercury in General Relativity, the Cosmological Constant and Jacobi's Inversion problem. *Class. Quantum Grav.* **20**, 4817-4835 (2003).
- Kraniotis G. V.: Precise relativistic orbits in Kerr and Kerr-(anti) de Sitter spacetimes. *Class. Quantum Grav.* **21**, 4743-4769 (2004).
- Maldacena J. M.: The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **2**, 231 (1998).
- Mann R. B.: Black holes of negative mass. *Class. Quant. Grav.* **14**, 2927 (1997).
- Mann R. B.: Lifshitz topological black holes. *JHEP* **06**, 075 (2009).
- Olivares M., Saavedra J., Leiva C. and Villanueva J. R.: Motion of Charged Particles on the Reissner-Nordström (Anti)-de Sitter Black Hole Spacetime. *Mod. Phys. Lett. A* **26**, 2923 (2011).
- Podolsky J.: The structure of the extreme Schwarzschild de Sitter Space-time. *Gen. Rel. Grav.* **31**, 1703 (1999).
- Stuchlík Z. and Calvani M.: Null geodesics in black hole metrics with non-zero cosmological constant. *Gen. Rel. Grav.* **23**, 507 (1991).
- Stuchlík Z. and Hledík S.: Properties of the Reissner-Nordström spacetimes with a nonzero cosmological constant. *Acta Phys. Slov.*, **52**, 363 (2002).
- Villanueva J. R., Saavedra J., Olivares M. and Cruz N.: Photons motion in charged Anti-de Sitter black holes. *Astrophys. Space Sci.* **344**, 437-446 (2013).